Kinetic Equations

Text of the Exercises

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Exercise 1

Recall that for any f measurable the *distribution function of* f is defined as $d_f(\gamma) := \left|\left\{x \in \mathbb{R}^d \middle| \; |f(x)| > \gamma\right\}\right|$. Prove that if $f \in L^p\left(\mathbb{R}^d\right)$ with $p \in [1, +\infty)$ then

$$\|f\|_{L^p(\mathbb{R}^d)} = \left(\int_0^{+\infty} p\gamma^{p-1} d_f(\gamma)\, d\gamma\right)^{\frac{1}{p}}. \tag{1}$$

Exercise 2

Let $m \geqslant 0$ and f a measurable function such that $f \geqslant 0$ and $f \in L^1 \cap L^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Define the measurable function $h : \mathbb{R} \times \mathbb{R}^3 \to [0, +\infty]$ as

$$h(t, x) := \int_{\mathbb{R}^3} f(x - tv, v)\, dv. \tag{2}$$

Show that there is a constant $C > 0$ such that

$$\|h(t, \cdot)\|_{L^{\frac{m+3}{3}}(\mathbb{R}^3)} \leqslant C \|f\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}^{\frac{m}{m+3}} \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^m f(x, v)\, dx dv\right)^{\frac{3}{m+3}}. \tag{3}$$

Exercise 3

Let $p \in [1, +\infty)$ and $f \in C\left(\mathbb{R}_+; L^p\left(\mathbb{R}^3 \times \mathbb{R}^3\right)\right) \cap L^\infty\left(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\right)$ a solution to the Vlasov-Poisson equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E(t, x) := -\nabla\left(\frac{1}{|x|} * \rho_t\right)(x), \\ \rho_t(x) := \int_{\mathbb{R}^3} f(t, x, v)\, dv, \\ f(0, x, v) = f_0(x, v). \end{cases} \tag{4}$$

Define $H(t)$ as

$$H(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f(t, x, v)\, dx dv + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_t(x)\, \rho_t(x')}{|x - x'|} dx dx'. \tag{5}$$

Prove that $H(t) = H(0)$.

Hint: Show that $\partial_t H(t) = 0$.